Non-monotonic dependence on disorder in biased diffusion on small-world networks

Damián H. Zanette

Consejo Nacional de Investigaciones Científicas y Técnicas - Centro Atómico Bariloche, 8400 San Carlos de Bariloche, Río Negro, Argentina

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Abstract. – We report numerical simulations of a strongly biased diffusion process on a one-dimensional substrate with directed shortcuts between randomly chosen sites, i.e. with a small-world-like structure. We find that, unlike many other dynamical phenomena on small-world networks, this process exhibits non-monotonic dependence on the density of shortcuts. Specifically, the diffusion time over a finite length is maximal at an intermediate density. This density scales with the length in a nontrivial manner, approaching zero as the length grows. Longer diffusion times for intermediate shortcut densities can be ascribed to the formation of cyclic paths where the diffusion process becomes occasionally trapped.

Complex networks are known to underly a wide class of natural and artificial systems, ranging from cellular tissues and human populations to language structures and computer webs [1, 2, 3, 4]. Among the various mathematical models for complex networks, small-world networks [1] capture a key property of actual biological and social systems, namely, the fact that sites with distant locations in space may be separated by only a few links over the network. Small-world networks, in fact, are constructed by adding randomly distributed shortcuts between distant sites in an initially ordered lattice, such that the average path length between any two sites results in a logarithmic dependence on the network size [5,6]. The effect of the underlying geometry on the dynamics of processes occurring on small-world networks has been analyzed with particular emphasis on propagation phenomena, such as diffusion and percolation [7, 8, 9, 10, 11, 12]. It has been shown that, in most cases, any finite density of shortcuts induces the behavior expected for random networks, where diffusion is a highly efficient transport mechanism and the usual linear scaling between mean square displacement and time breaks down. On the other hand, for slightly more complicated dynamics –where diffusion is combined with certain reaction-like processes– the random-network regime occurs only above a critical (finite) shortcut density [11, 12]. In all cases, however, the relevant quantities depend monotonically on the shortcut density. This is also the case for other kinds of phenomena on small-world networks, such as for Ising dynamics [5]. In this letter, we introduce a diffusion process on a small-world network with directed shortcuts where, in contrast, non-monotonic behavior is observed as the shortcut density is varied –i.e. as the

disorder of the substrate changes. Our numerical results reveal well-defined scaling properties in the limits of low and high disorder, and a nontrivial scaling law at the intermediate regime.

Consider a random walker which moves on a network constructed as follows. Starting from a one-dimensional array of N sites with nearest-neighbor connections, a link L_{ij} is established with probability p from each site i to a randomly chosen site j. On the average, pN shortcuts are thus created –typically, between otherwise distant sites of the array. The probability p gives the density of shortcuts, and measures the disorder or *randomness* of the resulting network. Sites are numbered correlatively, $i = 1$ to N, from one end of the array to the other. The random walker starts at the first site, $i = 1$. At each time step, it performs a forward jump of length k, from site i to site $i+k$. The length k is drawn at each step from a probability distribution $Q(k)$. If, however, a shortcut $L_{i+k,j}$ exists between sites $i + k$ and j, the final destination of the walker in the step under consideration is site j. Otherwise, it stays at $i+k$. The process stops when the random walker exits the system across the opposite end of the array, i.e. when from a site i a forward jump of length $k > N - i$ takes place.

In this model, consequently, transport is a combination of highly biased, unidirectional diffusion, whose statistical properties are determined by the distribution $Q(k)$, and strong mixing due to the shortcuts. The relative importance of both mechanisms is measured by the probability p. Note that, while the forward motion is here a stochastic process, mixing is deterministically defined by the links L_{ij}, i.e. by the disordered but frozen structure of the underlying network. Note also that shortcuts are directed: a link L_{ij} implies deterministic transport from i to j, but not in the opposite direction. The process is reminiscent of certain board games, such as "Snakes and Ladders" and its numerous variations. In this ancient Hindu game, forward motion represents reincarnation to higher forms of life in the way towards *nirvana*, threatened by the backward shortcuts which lead to inferior animal forms. In a more physical context, the model may be interpreted as a diffusion process in a one-dimensional medium under a strong external field, which induces forward motion, subject to the action of scattering events along well defined channels –perhaps due to localized defects or impurities.

The aim of the present analysis is to determine the typical duration T of the whole transport process as a function of the system length N and the degree of disorder p. In a single realization, the duration t is given by the number of time steps taken by the random walker to exit the system. The typical duration T must be defined as a suitable average of t over many realizations, as explained later. For convenience in the numerical implementation of the model, we take an exponential distribution of jump lengths, $Q(k) = q(1-q)^{k-1}$ ($0 < q < 1$, $k \geq 1$). The average jump length, $\langle k \rangle = q^{-1}$, gives the corresponding drift velocity, while the diffusion coefficient is proportional to the mean square dispersion $\langle k^2 \rangle - \langle k \rangle^2 = q^{-2}(1-q)$. For the numerical realizations, we choose $q = 1/2$.

We first study the distribution $f(t)$ of the duration t over series of numerical realizations. It can be easily realized that for large N and with $p = 0$, i.e. in the absence of shortcuts, $f(t)$ is approximately a Gaussian centered at qN and with width proportional to $q^{-1}(1-q)N$. Figure 1 shows numerical results for $f(t)$ and several values of p, with $N = 10^3$. Even for very small values of the randomness, $p \sim N^{-1}$ –where, on the average over realizations of the network, only one shortcut is added– the distribution develops a wide background over a large range of values of t. As p grows, this background rapidly replaces the Gaussian peak and widens, while a power-law large-t tail becomes apparent. For $p \approx 0.1$ the slope of this power-law tail attains its minimum (absolute) value, equal to -2 within numerical precision, and the distribution exhibits its broadest shape. For larger randomness, the tail recedes and becomes steeper. At $p = 1$, the slope has grown to approximately -2.6. This is a first indication of non-monotonic behavior as a function of the randomness.

A more compact description of the dependence of $f(t)$ with p is given by the typical time T

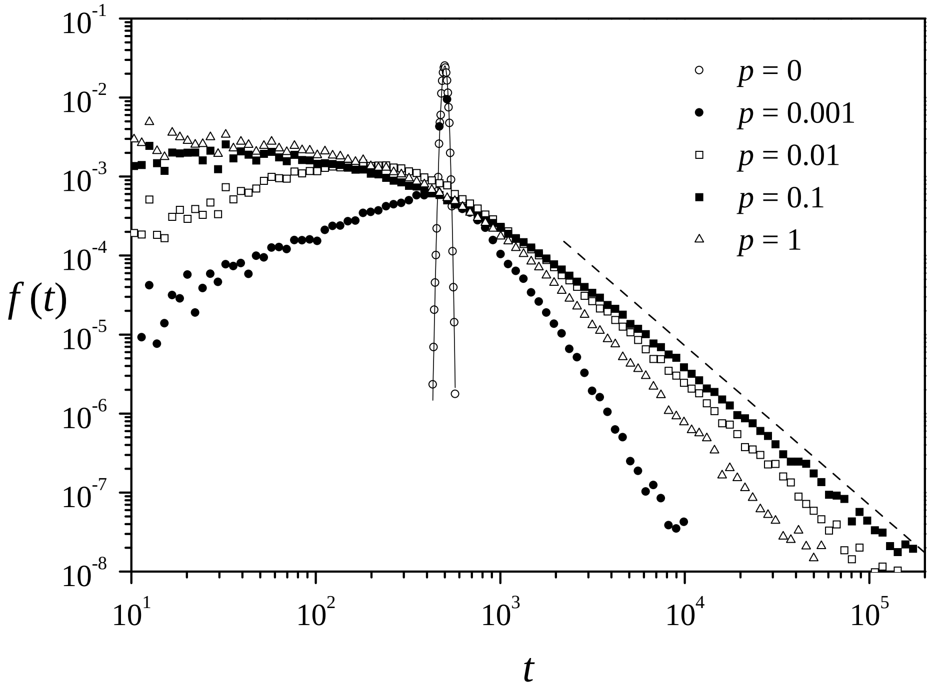

Fig. 1 – Normalized distribution of durations t of the transport process on a 10^3-site network, for different values of the randomness p. The full-line curve corresponds to a Gaussian fit for $p = 0$. The dashed straight line has slope -2. Note the non-monotonic dependence of the tail slope for $p > 0$.

taken by the random walker to exit the system. Such typical time, however, must be defined with caution. Indeed, a naive definition in terms of the average $\langle t \rangle = \int t f(t) \mathrm{d}t$ could fail to give numerical results with acceptable precision for the values p where the slope of the large-t tail of $f(t)$ is close to -2. For such randomness, large fluctuations between single realizations are expected in the value of $\langle t \rangle$. If the slope does in fact reach -2, the average $\langle t \rangle$ would directly be ill-defined. Consequently, we define

$$T = \left[\int_0^\infty t^\alpha f(t) \mathrm{d}t \right]^{1/\alpha}, \tag{1}$$

with $0 < \alpha < 1$. As far as $0 < \alpha < 1$, the choice is arbitrary. Below, we take $\alpha = 1/2$.

Figure 2 shows the typical time T as a function of randomness, for several system lengths ranging from $N = 3 \times 10^2$ to 10^4. Similar results, not shown in the figure, were obtained for up to $N = 10^5$. Non-monotonicity is apparent. While for $p \to 0$ we find the expected result $T \approx qN = N/2$ and for $p \to 1$ we have $T \approx 0.4N$, for intermediate randomness the ratio T/N reaches considerably higher values. As we discuss in more detail below, the randomness $p_{\max}$ at which T attains it maximum decreases with N, and the maximum itself grows. The two plots in fig. 2 reveal two well-defined scalings in the dependence of T on p. For $p \approx 1$ we have

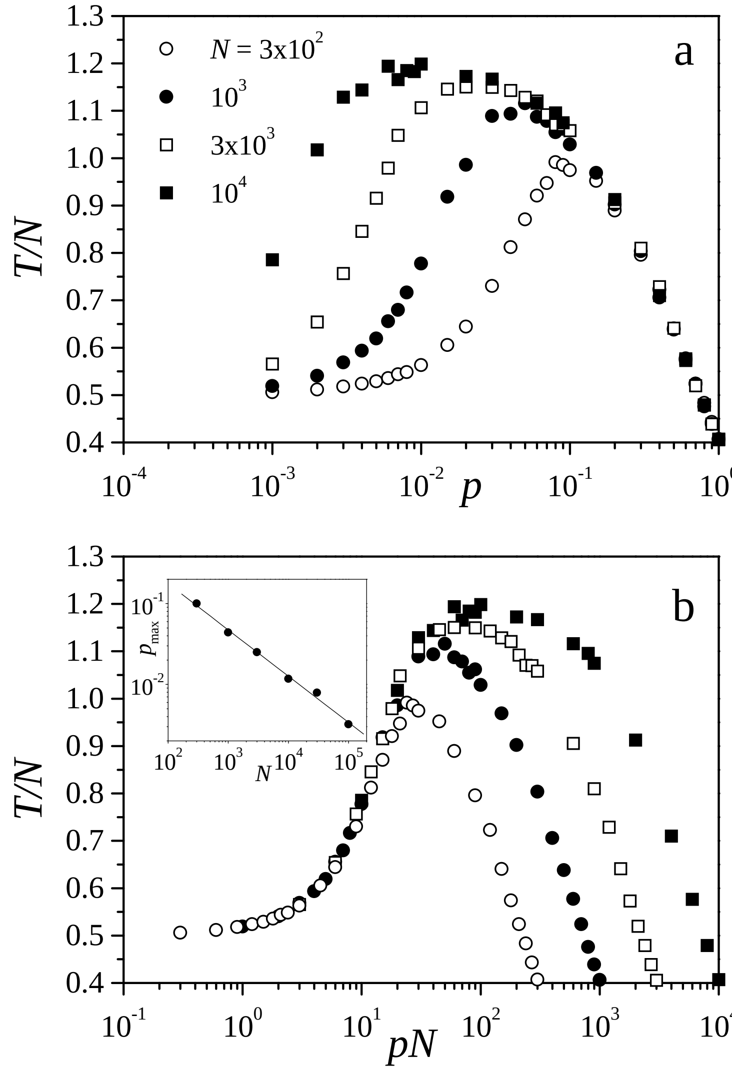

Fig. 2 – Typical duration of the transport process –defined as in eq. (1) with $\alpha = 1/2$– as a function of the randomness p, for several values of the length N. (a) T/N versus p, (b) T/N versus pN. Symbols are the same for both plots. The insert shows the position of the maximum, $p_{\max}$, as a function of N.

(see fig. 2a)

$$T = NF_1(p), \tag{2}$$

while for $p \approx 0$, we find (see fig. 2b)

$$T = NF_2(pN). \tag{3}$$

The scaling functions F_1 and F_2 are numerically found to be non-singular at the respective limits.

The scaling properties reported in eqs. (2) and (3) can be explained in qualitative terms with simple arguments. For sufficiently small randomness, only a few shortcuts are encountered in each realization of the process. Since shortcut ends are distributed at random, the average shortcut length is of order N. Therefore, the average duration qN at $p = 0$ is modified by a quantity of order N as p grows. Note that forward and backward shortcuts respectively diminish and increase the value of T. In any case, the resulting typical duration is proportional to N. Now, the number of different sites visited during a realization is also of order N, so that the probability of finding a shortcut is proportional to pN. Consequently, the typical duration T is expected to depend on p through the product pN only, as in eq. (3). Finally, since forward and backward shortcuts are equally probable, why does T increase with p? To answer this question, imagine a realization of the network with a single shortcut starting at site i. If the shortcut is forward, L_{ij} with $i < j$, it will be encountered by the random walker at most once in each realization, if the site i is visited. On the other hand, if the shortcut is backward, L_{ij} with $i > j$, there is a finite probability that the walker visits site i several times, remaining temporarily "trapped" into a cycle until it succeeds at skipping site i. Thus, the contribution of each backward shortcut to the typical duration T is effectively more important than the opposite contribution of each forward shortcut.

When the density of shortcuts is large, $p \approx 1$, there is a high probability for the random walker to encounter a shortcut at each step and therefore land at a random site in the network. The probability of exiting the system becomes large only when landing in the final end of the array, $i \approx N$. From there, diffusion can transport the walker outside the system, thus ending the process. The size of the region where the probability of abandoning the system by diffusion is above any fixed threshold is independent of N, so that the probability of reaching that region at any time step is of order N^{-1}. The typical duration of the process is therefore proportional to N. As for the dependence on p, T decreases as p grows simply because the probability of finding a shortcut leading to the final region of the array increases proportionally to the randomness. Note that the total number of shortcuts leading to (or starting from) a region of finite size is proportional to p and independent on N.

Since in the two regions $p \approx 0$ and $p \approx 1$ the scaling regimes of T as a function of p and N are different, the scaling properties in the intermediate region –where the duration is maximal– are not unambiguously defined. We have therefore recorded the randomness $p_{\max}$ at which the typical duration attains its maximum $T_{\max}$, for several system sizes up to $N = 10^5$. The insert of fig. 2b shows $p_{\max}$ as a function of N. Over the whole range, the position of the maximum is well approximated by a power law $p_{\max} \sim N^{-\gamma}$, with a nontrivial scaling exponent

$$\gamma = 0.57 \pm 0.02. \tag{4}$$

The maximum duration $T_{\max}$, on the other hand, does not show a well-defined behavior as a function of N. Our numerical precision does not make it possible to discern between a saturation towards $T_{\max}/N \approx 1.2$ or a slight growth of the ratio $T_{\max}/N$ with length. Discerning between these two possibilities would require realizations for substantially larger values of N.

In the intermediate regime, the effect of temporary trapping into cycles which contain backward shortcuts –discussed above in connection with the regime of small randomness– becomes dominant. Due to the fact that the probability of skipping any given site in the lattice is finite, the random walker cannot be indefinitely trapped in one of such cycles. We find however that repetition of the same cycle up to four or five times in a single realization is quite a common occurrence, which explains the relatively large durations observed in this regime. The role of cycles in the dynamics explains also the power-law tail in the distribution

of durations shown in fig. 1. In fact, it is known from other (deterministic) transport processes on lattices with sites connected by randomly distributed links –just like the shortcuts in our system– that cycle lengths are distributed according to a power law [13]. It is interesting to point out that the power-law distribution is observed even for $p = 1$ (see fig. 1), which suggests that in that limit the process is still governed by the existence of cycles. This complex ingredient prevents the analytical calculation of T for $p \approx 1$. In fact, a naive calculation of $f(t)$ using the arguments given above to explain the dependence of T on p and N in that limit, predicts an exponential distribution.

In summary, we have presented and numerically studied a stochastic transport process on a small-world-like one-dimensional substrate, which exhibits non-monotonic dependence on the substrate randomness p. This non-monotonicity is the result of competition between the formation of trapping cycles as the randomness grows and the overall improvement of transport efficiency due to the presence of shortcuts. Non-monotonicity naturally defines two separated regimes in the randomness domain, which are found to obey well-differentiated scaling laws in terms of p and the system size N. As for many other dynamical processes on small-world networks [5], the small-randomness regime is observed below $p \sim N^{-1}$ and thus collapses in a trivial way as N increases. In our system, however, we find a third, intermediate regime –where the process becomes extremal and non-monotonicity is therefore realized– characterized by a nontrivial scaling law, $p \sim N^{-\gamma}$ with $\gamma \approx 0.6$.

It is interesting to point out that, to our knowledge, the only previous report of non-monotonic dependence on randomness for dynamical processes on small-world networks also involves a directed network [14]. This leads to conjecture that this kind of non-monotonicity may be a common feature in directed small-world networks. Directed shortcuts in small-world networks –and, in general, in complex networks– have been scarcely treated in the literature, in spite of the fact that they should be essential to the mathematical description of certain non-physical (biological, social) interactions that have recently attracted much attention. Our analysis suggests that such kind of links can add considerable complexity to the involved processes.

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REFERENCES

[1] Watts D. J. and Strogatz S. H., *Nature*, **393** (1998) 440.
[2] Barabási A.-L. and Albert S., *Science*, **286** (1999) 509.
[3] Adamic L. A. *et al.*, *Science*, **287** (2000) 2115.
[4] Liljeros F. *et al.*, *Nature*, **411** (2001) 907.
[5] Barrat A. and Weigt M., *Eur. Phys. J. B*, **13** (2000) 547.
[6] Newman M. E. J., Moore C. and Watts D. J., *Phys. Rev. Lett.*, **84** (2000) 3201.
[7] Newman M. E. J. and Watts D. J., *Phys. Rev. E*, **60** (1999) 7332.
[8] Moore C., Newman M. E. J. and Watts D. J., *Phys. Rev. E*, **61** (2000) 5678.
[9] Manrubia S. C., Delgado J. and Luque B., *Europhys. Lett.*, **53** (2001) 693.
[10] Pastor-Satorras R. and Vespignani A., *Phys. Rev. E*, **63** (2001) 066117.
[11] Kuperman M. and Abramson G., *Phys. Rev. Lett.*, **86** (2001) 2909.
[12] Zanette D. H., *Phys. Rev. E*, **64** (2001) 050901R; **65** (2002) 041908.
[13] Zanette D. H., *J. Phys. A: Math. Gen.*, **30** (1997) L785.
[14] Sánchez A. D., López J. M. and Rodríguez M. A., *Phys. Rev. Lett*, **88** (2002) 048701.